# Category-Theoretic Reconstruction of Schemes from Categories of Reduced Schemes

Tomoki Yuji

September 30, 2021

# Today's Talk

Let $S$ be a scheme, $\blacklozenge/S$ a set of properties of $S$-schemes, and $\mathsf{Sch}_{\blacklozenge/S}$ the full subcategory of $\mathsf{Sch}_{/S}$ determined by the objects $X \in \mathsf{Sch}_{\blacklozenge/S}$ that satisfy every property of $\blacklozenge/S$.

In this talk, I will explain how to reconstruct $S$ from $\mathsf{Sch}_{\blacklozenge/S}$.

# Notations and Conventions

$S$ : Scheme

$\blacklozenge/S$ : a set of properties of $S$-schemes

$\mathsf{Sch}_{\blacklozenge/S}$ : $\begin{cases} \text{the full subcategory of } \mathsf{Sch}_{/S} \text{ determined by} \\ \text{the objects } X \in \mathsf{Sch}_{\blacklozenge/S} \text{ that satisfy every property of } \blacklozenge/S \end{cases}$

$\times, \lim$ : the fiber product, limit in Sch

$\times^{\blacklozenge}, \lim^{\blacklozenge}$ : the fiber product, limit in $\mathsf{Sch}_{\blacklozenge/S}$

In the present talk, we shall mainly be concerned with the properties

$$\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}.$$

# Previous Research

| | |
|---|---|
| Mochizuki 2004 | : $\blacklozenge/S =$ f.t.$/S$, $S$: locally Noetherian |
| | (+ log scheme version) |
| van Dobben de Bruyn 2019 | : $\blacklozenge = \varnothing$, $S$: arbitrary |
| Wakabayashi 2010 | : superscheme version of the case of Mochizuki |
| Anabelian Geometry | : $\blacklozenge/S =$ fét$/S$ |

These research and my research are motivated by anabelian geometry.

## Main Theorem

(1) $S$: locally Noetherian normal scheme, $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$ .
Then the following may be reconstructed category-theoretically from $\mathsf{Sch}_{\blacklozenge/S}$:

  (a) the structure of $T$ as a scheme (for every object $T \in \mathsf{Sch}_{\blacklozenge/S}$),
  (b) the structure of $f$ as a morphism of schemes (for every morphism $(f : X \to Y) \in \mathsf{Sch}_{\blacklozenge/S}$).

(2) $S, T$: quasi-separated,
$\blacklozenge, \lozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$ s.t. $\{\mathrm{qsep}, \mathrm{sep}\} \not\subset \blacklozenge, \{\mathrm{qsep}, \mathrm{sep}\} \not\subset \lozenge$
Then, $\mathsf{Sch}_{\blacklozenge/S} \cong \mathsf{Sch}_{\lozenge/T} \;\Rightarrow\; \blacklozenge = \lozenge$.

(3) $S, T$: locally Noetherian normal schemes, $\blacklozenge \subset \{\mathrm{red}, \mathrm{qcpt}, \mathrm{qsep}, \mathrm{sep}\}$.
Then, the following natural functor is equivalent:

$$\mathrm{Isom}(S, T) \to \mathbf{Isom}(\mathsf{Sch}_{\blacklozenge/T}, \mathsf{Sch}_{\blacklozenge/S})$$
$$f \mapsto f^*$$

# Outline

Since a scheme is constructed by

- the underlying set,
- the underlying topological space, and
- the structure sheaf,

to reconstruct a scheme,
it suffices to reconstruct these structures.

In the present talk,
I explain how to reconstruct the underlying sets, and
give category-theoretic characterizations of various properties used to reconstruct the underlying topological spaces and the structure sheaves.

# Remark: the Fiber Product in $\mathsf{Sch}_{\blacklozenge/S}$

**Lemma**

$f : Y \to X, g : Z \to X$ : morphisms in $\mathsf{Sch}_{\blacklozenge/S}$.
Suppose that either $f$ or $g$ is quasi-compact.

Then, the fiber product $Y \times_X^{\blacklozenge} Z$ in $\mathsf{Sch}_{\blacklozenge/S}$ exists,
and the following assertions hold:

If $\mathrm{red} \notin \blacklozenge$, then $Y \times_X^{\blacklozenge} Z \cong Y \times_X Z$.

If $\mathrm{red} \in \blacklozenge$, then $Y \times_X^{\blacklozenge} Z \cong (Y \times_X Z)_{\mathrm{red}}$.

In particular, $Y \times_X Z$ and $Y \times_X^{\blacklozenge} Z$ have same underlying top.

# An Idea to Reconstruct the Underlying Sets

**Observation**

A point $x \in X$ may be determined by

$$f : Y \to X \text{ s.t. } |Y|\text{: 1pt. set, and } \mathrm{Im}(f) = \{x\}.$$

Hence,

$$\text{giving a point of } X \iff$$
$$\text{giving a certein equivalence class of } f : Y \to X \text{ s.t. } |Y|\text{: 1pt. set.}$$

To reconstruct the underlying set, it suffices to characterize

one-pointed schemes (i.e., schemes whose underlying sets are 1pt. sets)

cat.-theoretically.

# Characterization of the One-Pointed Schemes

Let $X \in \mathsf{Sch}_{\blacklozenge/S}$.

**Characterization of the 1pt. Scheme**

$|X|$ is **not** $1$pt. set $\iff$
$\exists Y, Z \neq \varnothing$ , $\exists Y \to X, Z \to X$ s.t. $Y \times_X^{\blacklozenge} Z = \varnothing$

$\because)$
$X$ has two distinct pts. $x_1, x_2 \Rightarrow \mathrm{Spec}(k(x_1)) \times_X^{\blacklozenge} \mathrm{Spec}(k(x_2)) = \varnothing$.
$X$ satisfies the condition $\Rightarrow y \in Y, z \in Z$ determine two distinct pts. of $X$.

# Reconstruction of the Underlying Set 1

Let $X \in \mathsf{Sch}_{\blacklozenge/S}$. We define

$$\mathrm{Pt}_{\blacklozenge/S}(X) \overset{\mathsf{def}}{:=} \{(p_Z : Z \to X) \in \mathsf{Sch}_{\blacklozenge/S} \mid |Z|\text{: 1pt. set}\} \,/ \sim,$$

where

$$(p_Z : Z \to X) \sim (p_{Z'} : Z' \to X) \;\; :\overset{\mathsf{def}}{\Longleftrightarrow} \;\; Z \times^{\blacklozenge}_{p_Z, X, p_{Z'}} Z' \neq \varnothing.$$

**Reconstruction of the Underlying Set**

$\mathrm{Pt}_{\blacklozenge/S} : \mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Set}$ is naturally isomorphic to the functor $U^{\mathsf{Set}}_{\blacklozenge/S} : \mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Set}$.

Since the functor $\mathsf{Pt}_{\blacklozenge/S}$ is defined category-theoretically, the following corollary holds:

## Corollary

If $F : \mathsf{Sch}_{\blacklozenge/S} \to \mathsf{Sch}_{\lozenge/T}$ is an equivalence, then $U^{\mathsf{Set}}_{\blacklozenge/S} \cong U^{\mathsf{Set}}_{\lozenge/T} \circ F$.

$$\begin{array}{ccc} \mathsf{Sch}_{\blacklozenge/S} & \xrightarrow[\sim]{F} & \mathsf{Sch}_{\lozenge/T} \\ {\scriptstyle U^{\mathsf{Set}}_{\blacklozenge/S}}\downarrow & & \downarrow{\scriptstyle U^{\mathsf{Set}}_{\lozenge/T}} \\ \mathsf{Set} & = & \mathsf{Set}\,. \end{array}$$

# Regular Monomorphisms

$\mathcal{C}$: category, $(f : X \to Y) \in \mathcal{C}$.

## Definition

$f$ is a **regular monomorphism**

$:\overset{\text{def}}{\Longleftrightarrow}$ $\exists g, h : Y \to Z$, s.t., $f$ is the equalizer of $(g, h)$.

## Property of reg. mono. in $\mathsf{Sch}_{\blacklozenge/S}$

$S$: q.s., $(f : X \to Y) \in \mathsf{Sch}_{\blacklozenge/S}$: reg. mono. $\Rightarrow$ $f$: immersion.

$\because$) $f$: reg. mono. $\Rightarrow$ $f$: b.c. of the diagonal (details omitted).

## Corollary (Cat.-Theoretic Characterization of Red. Schemes)

$X \in \mathsf{Sch}_{\blacklozenge/S}$ is red. $\Longleftrightarrow$ [$f : Y \to X$: surj. reg. mono. $\Rightarrow$ $f$: isom.]

$\because$) a surj. reg. mono. is a surj. closed immersion.

# Closed Immersions

Closed immersions may be characterized as follows:

## Characterization of Closed Immersions

$S$: q.s., $(f : X \to Y) \in \mathsf{Sch}_{\blacklozenge/S}$.
$f$: closed immersion if and only if

- $f$: reg. mono.
- $\forall (T \to Y)$, the b.c. $X_{\blacklozenge,T} = X \times_Y^{\blacklozenge} T$ exists.
- $\forall (T \to Y)$, $\forall t \in T$: **closed pt.** s.t. $t \notin \mathrm{Im}(f_{\blacklozenge,T} : X_{\blacklozenge,T} \to T)$, $X_{\blacklozenge,T} \coprod \mathrm{Spec}(k(t)) \to T$: reg. mono.

Hence to give a cat.-theoretic characterization of closed immersions, it suffices to characterize the closed pt.
In particular, it suffices to characterize the relation $x_1 \rightsquigarrow x_2$.

# Strongly Local 1

$S$: q.s., $X \in \mathsf{Sch}_{\blacklozenge/S}$, $x_1, x_2 \in X$.

## Definition (Strongly Local)

$(X, x_1, x_2)$ is **strongly local** in $\mathsf{Sch}_{\blacklozenge/S}$ $:\overset{\text{def}}{\Longleftrightarrow}$

- $X$: connected.
- $\forall(f : Z \to X)$: reg. mono., $[x_1, x_2 \in \mathrm{Im}(f), \Rightarrow f$: isom.$]$.
- $\mathrm{Spec}(k(x_1)) \coprod \mathrm{Spec}(k(x_2)) \to X$: epi.
- $\mathrm{Spec}(k(x_1)) \to X$: reg. mono.
- $\forall(f : Z \to X)$: reg. mono., $[x_1 \notin \mathrm{Im}(f), Z \neq \varnothing \Rightarrow Z \coprod \mathrm{Spec}(k(x_1)) \to X$: **not** a reg. mono.$]$.

## Remark

The property that $(X, x_1, x_2)$ is strongly local is defined cat.-theoretically from the data $(\mathsf{Sch}_{\blacklozenge/S}, X, x_1, x_2)$.

# Strongly Local 2

$S$: q.s., $X \in \mathsf{Sch}_{\blacklozenge/S}, x_1, x_2 \in X$.

**Properties of Strongly Local Objects**

If $(X, x_1, x_2)$: strongly local, then

(1) $X \cong \mathrm{Spec}(\text{local domain})$

(2) One of $x_1, x_2$ is the closed pt., and the other is the generic pt.

In particular, $x_1 \rightsquigarrow x_2$ or $x_2 \rightsquigarrow x_1$.

Let $V = \mathrm{Spec}(\text{valuation ring})$, $v \in V$: closed pt., $\eta \in V$: generic pt.

**Proposition (Spec. of Valuation Rings are Strongly Local)**

$(V, v, \eta)$: strongly local.

# Cat.-Theoretic Characterization of "$x_1 \rightsquigarrow x_2$ or $x_2 \rightsquigarrow x_1$"

$S$: q.s., $X \in \mathsf{Sch}_{\blacklozenge/S}$, $x_1, x_2 \in X$.

**Cat.-Theoretic Characterization of "$x_1 \rightsquigarrow x_2$ or $x_2 \rightsquigarrow x_1$".**

"$x_1 \rightsquigarrow x_2$ or $x_2 \rightsquigarrow x_1$" $\iff$
$\exists Z \in \mathsf{Sch}_{\blacklozenge/S}, \exists z_1, z_2 \in Z, \exists (f : Z \to X) \in \mathsf{Sch}_{\blacklozenge/S}$, s.t.,
$(Z, z_1, z_2)$: str. loc., and $\{f(z_1), f(z_2)\} = \{x_1, x_2\}$.

By using the above characterization,
we can characterize the relation $x_1 \rightsquigarrow x_2$ (details omitted).

**Corollary**

(1) Closed immersions may be characterized cat.-theoretically.

(2) Underlying top. may be reconstructed cat.-theoretically.

In particular, top.-theoretic properties of schemes (or morphisms)
may be characterized cat.-theoretically
(ex: q.s., q.c., sep., irred., local ($\cong \mathrm{Spec}$(local ring)), open imm., univ. closed, etc.).

# Reconstruction of the Underlying Top.

(Similarly to the case of Set)
$\forall F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$, the following diagram commutes (up to isom.):

$$\begin{array}{ccc}
\mathsf{Sch}_{\blacklozenge/S} & \xrightarrow{\sim} & \mathsf{Sch}_{\lozenge/T} \\
{\scriptstyle U^{\mathsf{Top}}_{\blacklozenge/S}} \downarrow & & \downarrow {\scriptstyle U^{\mathsf{Top}}_{\lozenge/T}} \\
\mathsf{Top} & = & \mathsf{Top}
\end{array}$$

# An Observation

To reconstruct the structure sheaf of $X \in \mathsf{Sch}_{\blacklozenge/S}$,
it suffices to characterize the ring scheme $\mathbb{A}^1_X \to X$ cat.-theoretically.

Since $\mathbb{A}^1$ is f.p. over a base scheme,
we want to get a cat.-theoretic characterization of f.p. morphisms.

**Idea**

f.p.$/S \risingdotseq$ a "compact object" in $\mathsf{Sch}^{\mathrm{op}}_{/S}$

More precisely,
$X \to S$: f.p. $\iff$
$\forall (V_\lambda, f_{\lambda\mu})_{\lambda\in\Lambda}$: diagram in $\mathsf{Sch}_{/S}$ s.t.
$\Lambda$: cofiltered, $V_\lambda$: affine,
the following natural map is surj. :

$$\varphi : \operatorname*{colim}_{\lambda\in\Lambda^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Sch}_{/S}}(V_\lambda, X) \to \mathrm{Hom}_{\mathsf{Sch}_{/S}}(\lim_{\lambda\in\Lambda}{}^{\blacklozenge} V_\lambda, X).$$

# Locally of Finite Presentation Morphisms 1

$S$: q.s., $(f : X \to Y) \in \mathsf{Sch}_{\blacklozenge/S}$, $x \in X$.

## Proposition

$f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$: essentially of finite presentation $\iff$
$\forall (V_\lambda, f_{\lambda\mu})_{\lambda\in\Lambda}$: diagram in $\mathsf{Sch}_{\blacklozenge/Y}$ s.t.
$\Lambda$: cofiltered, $V_\lambda$: local, $f_{\lambda\mu}(\text{closed pt.}) = f(x)$,
the following natural map is surjective :

$$\varphi : \operatorname*{colim}_{\lambda\in\Lambda^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/Y}}(V_\lambda, X) \to \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/Y}}(\lim_{\lambda\in\Lambda}{}^{\blacklozenge} V_\lambda, X).$$

$\because$) f.p. schemes (over $Y$) are cpt. objects in $\mathsf{Sch}_{/Y}$ (details omitted).

# Locally of Finite Presentation Morphisms 2

$S$: q.s., $(f : X \to Y) \in \mathsf{Sch}_{\blacklozenge/S}$.

## Cat.-Theoretic Characterization of Loc.F.P. Morphisms

$f$: loc. of f.p. $\iff$

- $\forall x \in X$, $f_x^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$: essentially of finite presentation.
- $\forall (Z \to Y)$, $\forall z \in Z$, the following natural map is bijective :

$$\varphi_{z,X} : \operatorname*{colim}_{W \in I_Z(z)^{\mathrm{op}}} \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/Y}}(W, X) \to \mathrm{Hom}_{\mathsf{Sch}_{\blacklozenge/Y}}(\lim_{W \in I_Z(z)}{}^{\blacklozenge} W, X),$$

where $I_Z(z) :\overset{\mathsf{def}}{=} \{i_W : W \to Z \mid i_W\text{: open imm., } z \in \mathrm{Im}(i_W)\}$.

$\because$) f.p. schemes (over $Y$) are cpt. objects in $\mathsf{Sch}_{/Y}$ (details composited).

# List of Cat.-Theoretic Properties

$S$: q.s.
$\forall X \in \mathsf{Sch}_{\blacklozenge/S}$, $|X|$ has been reconstructed cat.-theoretically, and

the following scheme-theoretic properties have been characterized cat.-theoretically:

- red., irred., integral, q.c., $\cong \mathrm{Spec}$(local ring), $\cong \mathrm{Spec}$(field).
- q.c., q.s., sep., imm., closed imm., open imm., loc. of f.p., f.p., f.p. $+$ proper ($=$ sep.$+$ f.p.$+$ univ. closed).

The following properties have not given yet cat.-theoretic characterizations:

flat, smooth, étale, etc.

# An Idea to Reconstruct the Structure Sheaves

To reconstruct the structure sheaf of $X \in \mathsf{Sch}_{\blacklozenge/S}$,
it suffices to characterize the ring scheme $\mathbb{A}^1_X \to X$ cat.-theoretically.
Since $\mathbb{A}^1_X = \mathbb{P}^1_X \setminus \{\infty\}$,
it suffices to characterize $\mathbb{P}^1_X \to X$ cat.-theoretically.

**What to Do**

Give a cat.-theoretic characterizaion of $\mathbb{P}^1$.

# The Case where $X = \mathrm{Spec}(k)$

$$\mathbb{P}^1_k \iff \begin{cases} \bullet \text{ proper over } \mathrm{Spec}(k) \\ \bullet \text{ the residue field of the generic pt. } \cong k(t) \\ \bullet \text{ "Closest" to } \mathrm{Spec}(k(t)) \end{cases}$$

$\therefore$ it suffices to characterize $\mathrm{Spec}(k(t)) \to \mathrm{Spec}(k)$.

Idea: Lüroth's theorem.

## Cat.-Theoretic Characterization of $k(t)/k$

$f : Y \to \mathrm{Spec}(k)$: isom. to $\mathrm{Spec}(k(t)) \to \mathrm{Spec}(k)$ over $\mathrm{Spec}(k)$ $\iff$

- $\exists K$ : field , $Y \cong \mathrm{Spec}(K)$
- $f$: not f.p. ($\Leftrightarrow$ $K/k$: not a finite extension)
- $k \subsetneq \forall L \subset K$, $\exists$ isom. $K \cong L$ over $k$ (Lüroth's theorem).

$\rightsquigarrow$ We obtain a cat.-theoretic characterization of $\mathbb{P}^1_k \to \mathrm{Spec}(k)$.

# An Idea in the Case of General Base Scheme 1

To characterize $\mathbb{P}^1_X \to X$,
it suffices to characterize $\mathbb{P}^1_S \in \mathsf{Sch}_{\blacklozenge/S}$.
Since

$$\mathbb{P}^1_S \iff \mathbb{P}^1\text{-bundle}/S + \exists\ 3 \text{ sections } s_1, s_2, s_3 \text{ s.t. } s_i \cap s_j = \varnothing, (i \neq j),$$

it suffices to characterize the $\mathbb{P}^1$-bundle over $S$.
$\rightsquigarrow$ $\mathbb{P}^1$-bundle $\Rightarrow$ **each fiber is** $\mathbb{P}^1$.

## Remark

- If $\mathrm{red} \in \blacklozenge$, then cat.-theoretic fiber $\ncong$ scheme-theoretic fiber.
- A generic fiber may be presented by a limit of open immersions.
  $\rightsquigarrow$ cat.-theoretic generic fiber $\cong$ scheme-theoretic generic fiber.

# An Idea in the Case of General Base Scheme 2

$\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ has a ring scheme structure:

**Observation**

1-dim ring scheme $\risingdotseq \mathbb{A}^1$ ??

**Lemma (♠♠♠)**

$R$: DVR, $V :\stackrel{\text{def}}{=} \operatorname{Spec} R$, $K :\stackrel{\text{def}}{=} \operatorname{Frac}(R)$ $f : X \to V$: flat ring scheme $/V$.
If $f$ satisfies the following conditions, then $X \cong \mathbb{A}^1_V$ and $f$ is the proj.:

- The special fiber of $f$ is connected and 1-dim.
- The generic fiber of $f$ is $\mathbb{A}^1_K$.

Without connectedness of the special fiber, there is a counterexample:
$\operatorname{Spec}(R[x, (x^{p^2} - x^p)/\pi])$.

# The Case of General Base Scheme

**Theorem**

$S$: locally Noetherian normal, $\lozenge = \blacklozenge \cup \{\mathrm{red}\}$, $(f : X \to S) \in \mathsf{Sch}_{\blacklozenge/S}$.
$f$ is isom. to $\mathbb{P}^1_S \to S \iff f$ satisfies the following conditions:

(1) $f$: f.p. proper.

(2) $\forall s \in S$, $f^{-1}(s)_{\mathrm{red}} \cong \mathbb{P}^1_{k(s)}$.

(3) $\forall$generic pt. $\eta \in S$, $f^{-1}(\eta) \cong \mathbb{P}^1_{k(\eta)}$.

(4) $\exists s_0, s_1, s_\infty$: sections of $f$ s.t. $s_i \cap s_j = \varnothing, (i \neq j)$.

(5) $\forall i = 0, 1, \infty$, $\exists$ a ring structure on $X \setminus s_i$ over $S$ in $\mathsf{Sch}_{\lozenge/S}$ s.t. $s_j$: add. unit, $s_k$: mult. unit, and $\{i, j, k\} = \{0, 1, \infty\}$.

(6) $(g : Y \to S) \in \mathsf{Sch}_{\blacklozenge/S}$, $t_0, t_1, t_\infty$: sections, s.t. satisfy (1),...,(5), $\Rightarrow \exists ! h : X \to Y$: closed imm. s.t. $\forall i = 0, 1, \infty, f = g \circ h, h \circ s_i = t_i$.

# Proof

If $\mathbb{P}^1$ satisfies (6), then by the uniqueness of (6), "$\Leftarrow$": ok.
$\therefore$ It suffices to prove "$\Rightarrow$" (i.e., $\mathbb{P}^1_S$ satisfies (6)).
$Y$: satisfies (1),...,(5). We define

$$C : \mathsf{Sch}_{/S}^{\mathrm{op}} \to \mathsf{Set},$$
$$(T \to S) \mapsto \left\{ i : \mathbb{P}^1_T \to Y_T \;\middle|\; \begin{array}{c} i\text{: closed imm.,} \\ 0, 1, \infty \mapsto t_0, t_1, t_\infty \end{array} \right\}.$$

Then,

- $C$: algebraic space $/S$.
- by (2), each fiber of $C \to S$ is a 1-pt. set.
- by (3), $C \to S$ is birational.

**What to Prove**

$C(S)$: 1-pt. set.

# Proof

$W :\overset{\text{def}}{=} \mathrm{Spec}(\mathsf{DVR})$.

$\forall (W \to S)$, $(Y_W)_{\mathrm{red}} \setminus t_{i,W}$: flat ring scheme $/W$.
$\therefore$ $\forall W$, $C(W)$: 1-pt. set ($\Rightarrow C$ is a scheme).
By lemma (♠♠♠) and a valuative criterion,
$C \to S$: proper birat. bij. ($\Rightarrow$ finite).
Since $S$ is normal, $C_{\mathrm{red}} \cong S$ (by ZMT).

In particular, $C(S)$: 1-pt. set.
$\rightsquigarrow$ Q.E.D.

(Similarly to the case of Set, Top)
$\forall F : \mathsf{Sch}_{\blacklozenge/S} \xrightarrow{\sim} \mathsf{Sch}_{\lozenge/T}$: equiv.,
the following diagram commutes (up to isom.):

$$\begin{array}{ccc} \mathsf{Sch}_{\blacklozenge/S} & \xrightarrow{\sim} & \mathsf{Sch}_{\lozenge/T} \\ {\scriptstyle U^{\mathsf{Sch}}_{\blacklozenge/S}}\downarrow & & \downarrow{\scriptstyle U^{\mathsf{Sch}}_{\lozenge/T}} \\ \mathsf{Sch} & = & \mathsf{Sch}. \end{array}$$

Moreover, the following equiv. holds:

$$\mathrm{Isom}(S, T) \xrightarrow{\sim} \mathbf{Isom}(\mathsf{Sch}_{\blacklozenge/T}, \mathsf{Sch}_{\blacklozenge/S}).$$

# Related Works

I also confirmed that the following problem has been solved:

- Reconstructing a Noetherian scheme $S$ from the category of finite $S$-schemes.

Since we may consider many properties of schemes,
there are many cat.-theoretic reconstruction problems.

Thank you for your attention.